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The superparticle propagator

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Abstract. We carry out path integral quantisation of the superparticle action in D-dimensional spacetime using the FV theorem. Having fixed the gauge, we obtain a closed form for the propagator by restricting the region of integration to a single covering of moduli space. We present specific calculations for $D=2$ and then generalise to $D=2$, 3, 4 mod 8.

1. Introduction

An understanding of the covariant quantisation of the superparticle should point the way to methods for quantising the superstring (where the need to explicitly separate the first- and second-class components of the fermionic constraints causes problems). Indeed Nissimov *et al* have applied harmonic superspace techniques, developed in quantising the superparticle [1], to covariantly quantise the superstring [2]. Here we present a covariant quantisation scheme for the superparticle that does not require a consideration of harmonic superspaces. Our path integral approach to the superparticle is similar to that of Hori *et al* [3]. We separate the fermionic constraints into non-covariant components and quantise the theory using the BFV path integral (see [4]). With a non-covariant gauge choice we obtain a manifestly covariant path integral and show that our region of integration reduces to a single cover of Teichmuller space. We then obtain the propagator by applying Govaerts' argument [5] and restricting the region of integration to a single cover of moduli space.

Our superparticle action is [6]

$$S=\int_{\tau_i}^{\tau_f}\mathrm{d}\tau L=\int_{\tau_i}^{\tau_f}\mathrm{d}\tau\frac{1}{2e}(\dot{x}^\mu+\mathrm{i}\dot{\theta}_a\Gamma^\mu_{ab}C_{bc}\bar{\theta}_c)^2 \qquad (1)$$

where e is a world line einbein, $\bar{\theta}_a$ is the Dirac adjoint of a Majorana fermion θ_a, C_{ab} is the charge conjugation matrix and Γ^μ_{ab} are the Dirac matrices. The action (1) is obtained from a more general action [7] by implementing the Majorana condition, $\theta_a=C_{ab}\bar{\theta}_b$. Our calculations are therefore applicable to spacetimes of dimension 2, 3 and 4 modulo 8. Our dynamical variables are taken to be x^μ, p^μ, e, π_e, $\bar{\theta}_a$ and ρ_a where we define

$$p^\mu=\frac{\partial L}{\partial\dot{x}_\mu}\qquad\rho_a=\frac{\partial L}{\partial\dot{\theta}_a}\qquad\pi_e=\frac{\partial L}{\partial\dot{e}}. \qquad (2)$$

Application of Dirac's constraint algorithm [8] to (1) gives us the constraints T_i, $i=1,2,3$:

$$T_1=\pi_e\approx0\qquad T_2=p^\mu p_\mu\approx0\qquad T_{3a}=\rho_a-\mathrm{i}\not{p}_{ab}C_{bc}\bar{\theta}_c\approx0. \qquad (3)$$

The Hamiltonian on the surface of constraint, H_0, and the total Hamiltonian, H_T, are

$$H_0 = 0 \qquad H_T = \tfrac{1}{2} e p_\mu p^\mu + \lambda_i T_i \qquad i = 1, 2, 3 \tag{4}$$

where the λ_i are Lagrange multipliers for the constraints T_i. Defining the Poisson brackets

$$\{x^\mu, p_\nu\}_{PB} = \delta^\mu_\nu \qquad \{e, \pi_e\}_{PB} = 1 \qquad \{\bar{\theta}_a, \rho_b\} = -\delta_{ab} \tag{5}$$

as the only non-zero brackets, we can form the Poisson brackets between the constraints:

$$\{T_i, T_j\}_{PB} = 0 \qquad i = 1, 2 \quad j = 1, 2, 3 \qquad \{T_{3a}, T_{3b}\}_{PB} = 2\mathrm{i}\not{p}_{bc} C_{ca}. \tag{6}$$

Here we see the Poisson bracket of two supersymmetry operators (the fermionic constraints T_{3a}) gives a spacetime translation operator, as should be the case for a supersymmetric theory (see [9]). Now, the rank of $\not{p}C$ on the surface of constraint is equal to the number of second-class components of T_{3a} since second-class constraints are defined to be those whose Poisson brackets with each other do not vanish, even weakly. In order to apply the FV theorem, the first- and second-class components of T_{3a} must be separated. The explicit separation of these components is relatively straightforward in two dimensions, so we use the example of $D = 2$ to demonstrate our method.

2. The propagator for $d = 2$

We choose the following representation for the Γ^μ_{ab} matrices in 2D spacetime:

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Gamma^1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \tag{7}$$

such that

$$\Gamma^3 = \Gamma^0 \Gamma^1 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = C_{ab}. \tag{8}$$

Our metric is

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

With this choice of Dirac matrices, elementary row operations give

$$\mathrm{rank}(\not{p}C) \approx \tfrac{1}{2} \dim(\not{p}C) = 1. \tag{10}$$

By a theorem given by Sohnius [9] we can show that this result is actually independent of our choice of the Γ^μ. So T_{3a} has one first-class and one second-class component. Labelling the components of T_{3a} as T_α and T_β, we find that $-T_\alpha + T_\beta$ is second class and

$$(\mathrm{i}T_\alpha - T_\beta)\Theta(p^0 p^1) + (T_\alpha - \mathrm{i}T_\beta)\Theta(-p^0 p^1) \tag{11}$$

is first class. We use the Heaviside functions, $\Theta(p)$, to ensure that our separation of T_{3a} into first- and second-class components is valid on both branches of the mass shell; $p^0 = p^1$ and $p^0 = -p^1$. Let us then relabel our constraints:

$$T_1 = \pi_e \approx 0 \tag{12a}$$

$$T_2 = p^\mu p_\mu \approx 0 \tag{12b}$$

$$T_3 = (\mathrm{i}T_\alpha - T_\beta)\Theta(p^0p^1) + (T_\alpha - \mathrm{i}T_\beta)\Theta(-p^0p^1) \approx 0 \tag{12c}$$

$$T_4 = -T_\alpha + T_\beta \approx 0 \tag{12d}$$

such that we have the Poisson brackets

$$\{T_i, T_j\}_{\mathrm{PB}} = 0 \qquad i = 1, 2 \quad j = 1, 2, 3, 4 \tag{13a}$$

$$\{T_3, T_3\}_{\mathrm{PB}} = 4(p^0 - p^1)\Theta(p^0p^1) - 4(p^0 + p^1)\Theta(-p^0p^1) \approx 0 \tag{13b}$$

$$\{T_3, T_4\}_{\mathrm{PB}} = 2(\mathrm{i} - 1)[(p^0 - p^1)\Theta(p^0p^1) - (p^0 + p^1)\Theta(-p^0p^1)] \approx 0 \tag{13c}$$

$$\{T_4, T_4\}_{\mathrm{PB}} = 4\mathrm{i}p^1 \not\approx 0. \tag{13d}$$

It is interesting to note that the fermionic second-class constraint T_4 is its own canonical conjugate, so we have a theory with a single second-class constraint. Each second-class constraint which occurs in a theory implies one of the theory's phase space variables is redundant. The $D = 2$ superparticle is therefore a theory with an odd-dimensional phase space. This raises certain interpretational difficulties which are briefly mentioned below. Due to the presence of the second-class constraint, T_4, we must set up the Dirac bracket and, in obtaining the quantum theory, work with this rather than the Poisson bracket. For two functions A and B the Dirac bracket is

$$\{A, B\}_{\mathrm{DB}} = \{A, B\}_{\mathrm{PB}} - \{A, T_4\}_{\mathrm{PB}}(4\mathrm{i}p^1)^{-1}\{T_4, B\}_{\mathrm{PB}}. \tag{14}$$

With Poisson brackets replaced by Dirac brackets, $T_4 = 0$ is taken as a strong equation rather than a weak constraint equation. We now write the total Hamiltonian as

$$H_{\mathrm{T}} = \lambda_i T_i \qquad i = 1, 2, 3 \tag{15}$$

with suitably redefined λ_2. Notice that, since $H_0 = 0$, our equations of motion will be invariant under reparameterisations of the time evolution parameter $\tau \to f(\tau)$. Consider some dynamical variable $g(\tau)$:

$$\begin{aligned} \frac{\mathrm{d}g(\tau)}{\mathrm{d}\tau} &= \{g(\tau), H_T\}_{\mathrm{PB}} \approx \lambda_i(\tau)\{g(\tau), T_i\}_{\mathrm{PB}} \\ \frac{\mathrm{d}g(\tau)}{\mathrm{d}f(\tau)} &= \frac{\mathrm{d}g(\tau)}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}f(\tau)} \approx \lambda_i(\tau)\frac{d\tau}{\mathrm{d}f(\tau)}\{g(\tau), T_i\}_{\mathrm{PB}} = \lambda_i'(f(\tau))\{g(f(\tau)), T_i\}_{\mathrm{PB}}. \end{aligned} \tag{16}$$

Notice also that the Lagrange multipliers $\lambda_i(\tau)$ transform under reparameterisations as einbein fields.

We now quantise the superparticle by following the prescription given in [4]. Extend the phase space by allowing the multipliers λ_i ($i = 1, 2, 3$) to become dynamically active. Introduce momenta, v_i, conjugate to the λ_i and constrain them to be zero; $v_i \approx 0$. Introduce ghost fields η_i and conjugate momenta $\bar{\eta}_i$ of opposite statistics to the T_i and similarly ghosts and momenta ζ_i and $\bar{\zeta}_i$ for the v_i. So we have the superlarge phase space

$$\{x^\mu, p^\mu, e, \pi_e, \bar{\theta}_a, \rho_a, \lambda_i, v_i, \eta_i, \bar{\eta}_i, \zeta_i, \bar{\zeta}_i\} \qquad i = 1, 2, 3 \tag{17}$$

with Poisson brackets

$$\{\lambda_i, v_j\}_{\mathrm{PB}} = (-1)^{\varepsilon_i}\delta_{ij} \qquad \{\eta_i, \bar{\eta}_j\}_{\mathrm{PB}} = (-1)^{\varepsilon_i}\delta_{ij} \qquad \{\zeta_i, \bar{\zeta}_j\}_{\mathrm{PB}} = (-1)^{\varepsilon_i}\delta_{ij} \tag{18}$$

where ε_i is the Grassman parity of the variable with subscript i. Consideration of the algebra of the constraints (v_i, T_i) $(i = 1, 2, 3)$ under the Dirac bracket operation allows us to construct the BRST generator Ω:

$$\Omega = v_i \zeta_i + T_i \eta_i + \bar{\eta}_2 \eta_3 \eta_3 / p^1. \tag{19}$$

Notice that Ω is nilpotent, $\{\Omega, \Omega\}_{\mathrm{DB}} = 0$.

With the following BRST invariant boundary conditions

$$\begin{aligned} &x^\mu(\tau_i) = x^\mu_I \qquad x^\mu(\tau_f) = x^\mu_F \qquad \eta_i(\tau_i) = 0 = \eta_i(\tau_f) \\ &\bar{\theta}_a(\tau_i) = \bar{\theta}^I_a \qquad \bar{\theta}_a(\tau_f) = \bar{\theta}^F_a \qquad \bar{\zeta}_i(\tau_i) = 0 = \bar{\zeta}_i(\tau_f) \\ &e(\tau_i) = e_I = e(\tau_f) = e_F \qquad v_i(\tau_i) = 0 = v_i(\tau_f) \end{aligned} \tag{20}$$

the BFV path integral [10] for the superparticle is then

$$\begin{aligned} Z_\Psi \sim \int &[\mathrm{d}x^\mu][\mathrm{d}p^\mu][\mathrm{d}e][\mathrm{d}\pi_e][\mathrm{d}\bar{\theta}_a][\mathrm{d}\rho_a][\mathrm{d}\lambda_i][\mathrm{d}v_i][\mathrm{d}\eta_i][\mathrm{d}\bar{\eta}_i][\mathrm{d}\zeta_i][\mathrm{d}\bar{\zeta}_i]|\det\{T_4, T_4\}_{\mathrm{PB}}|^{1/2} \\ &\times \delta[T_4] \exp\left(\mathrm{i}\int_{t_i}^{\tau_f} \mathrm{d}\tau (\dot{x}_\mu p^\mu + \dot{e}\pi_e + \dot{\bar{\theta}}_a \rho_a + \dot{\lambda}_i v_i + \dot{\eta}_i \bar{\eta}_i + \dot{\zeta}_i \bar{\zeta}_i - \{\Psi, \Omega\}_{\mathrm{DB}})\right) \\ &i = 1, 2, 3 \end{aligned} \tag{21}$$

where Ψ is a ghost number -1 gauge Fermion. Notice that we can write

$$\delta[T_4] \sim \int [\mathrm{d}\phi] \exp\left(\mathrm{i}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \phi T_4\right) \tag{22}$$

where ϕ is some Fermionic variable, and

$$|4\mathrm{i}p^1|^{1/2} = \int [\mathrm{d}\xi] \exp\left(\mathrm{i}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \frac{\pi\xi^2}{4|p^1|}\right) \tag{23}$$

where ξ is Bosonic. The interpretation of (21) in terms of infinite-dimensional integrals by taking time slices is given in the appendix.

The Fradkin–Vilkovisky theorem tells us that for suitably chosen BRST invariant boundary conditions Z_Ψ is independent of the choice of Ψ. Govaerts [5] explains that the FV theorem cannot be quite correct. Govaerts restates the FV theorem as follows: 'the BFV path integral does not specifically depend on a given gauge fixing function, Ψ, but only on its gauge-equivalence class.' A gauge-equivalence class of gauge fixing functions consists of all the Ψ that determine the same covering of Teichmuller space. Govaerts describes a gauge fixing function Ψ as good if it leads us to a single covering of Teichmuller space. Nelson [11] gives the definition of Teichmuller space, in the context of string and point particle theories, as

$$\text{Teich} = \frac{\{\text{metrics}\}}{\{\text{rescalings}\} \times \{\text{connected diffeomorphisms}\}}. \tag{24}$$

Since our multipliers $\lambda_i(\tau)$ transform under world line diffeomorphisms $\tau \to f(\tau)$ according to the einbein transformation rule

$$\lambda(\tau) \to \lambda_i'(f(\tau)) = \lambda_i(\tau) \frac{\mathrm{d}\tau}{\mathrm{d}f(\tau)} \tag{25}$$

and since the superparticle has no Weyl rescaling symmetry (see [6]), the Teichmuller space for the superparticle is

$$\text{Teich} = \frac{\{\lambda_i(\tau)\}}{\{\text{connected diffeomorphisms}\}}. \tag{26}$$

The parameters $c_i = \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \lambda_i(\tau)$ are invariant under the connected diffeomorphisms $\tau \to f(\tau)$ and so can be used as Teichmuller parameters for the superparticle. Choosing a gauge for any theory for which $H_0 = 0$ is equivalent to choosing a specific time evolution parameter, $f(\tau)$, and so, by (25), is equivalent to choosing a set of multipliers, $\lambda_i(\tau)$. A gauge for which $\{\Psi, \Omega\}_{\mathrm{DB}}$ is not a function of the v_i yields a functional delta function $\delta[\dot{\lambda}_i]$, on integrating over the v_i. Such a gauge is called a proper time gauge. Working in a gauge of this sort we have $c_i = \lambda_i^0(\tau_f - \tau_i)$. As $(\tau_f - \tau_i)$ goes from $-\infty$ to ∞ so does c_i. With λ_3^0 an element of a Grassman algebra, G, our Teichmuller space can be seen to be $\mathbb{R}^2 \times g$, $g = \{(\tau_f - \tau_i)\lambda_3^0\colon\ \lambda_3^0 \in G,\ (\tau_f - \tau_i) \in \mathbb{R}\}$.

We consider the gauge choice

$$\Psi = \bar{\eta}_1\lambda_1 + \bar{\eta}_2\lambda_2 - \frac{\phi T_4 \bar{\eta}_2}{T_2} + \phi F(\lambda_3)\bar{\zeta}_1 - \frac{\xi^2 \bar{\eta}_1}{\pi_e}\left(\frac{\pi}{4|p^1|} - v_1^2\right) \tag{27}$$

where $F(\lambda_3)$ is a Fermionic function, and make the changes of variables

$$\bar{\zeta}_1 \to \gamma\bar{\zeta}_1' \qquad \zeta_1 \to \zeta_1'/\gamma \qquad \bar{\eta}_1 \to \gamma\bar{\eta}_1' \qquad \eta_1 \to \eta_1'/\gamma \tag{28}$$

which have a super-Jacobian equal to 1 and then let $\gamma \to 0$ once Ψ has been substituted into Z_Ψ. Integrating over ϕ and ξ and dropping the primes on the transformed variables, we have

$$Z \sim \int [\mathrm{d}x^\mu][\mathrm{d}p^\mu][\mathrm{d}e][\mathrm{d}\pi_e][\mathrm{d}\bar{\theta}_a][\mathrm{d}\rho_a][\mathrm{d}\lambda_i][dv_i][\mathrm{d}\eta_i][\mathrm{d}\bar{\eta}_i][\mathrm{d}\zeta_i][\mathrm{d}\bar{\zeta}_i]\delta[F(\lambda_3)]$$
$$\times \exp\left(\mathrm{i}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau(\dot{x}_\mu p^\mu + \dot{e}\pi_e + \dot{\bar{\theta}}_a\rho_a + \dot{\lambda}_i v_i + \dot{\eta}_i\bar{\eta}_i + \dot{\zeta}_i\bar{\zeta}_i\right.$$
$$\left. - \bar{\eta}_1\zeta_1 - \bar{\eta}_2\zeta_2 + \lambda_2 p_\mu p^\mu + \lambda_1\pi_e)\right). \tag{29}$$

Notice that (29) is covariant; the steps implicit in (22), (23) and the gauge choice (27) are equivalent to the procedure of 'covariantisation' described in [3].

We evaluate the integrals in (29) by taking 'time slices' (see, for example, [12]) as defined in the appendix or we could use [13] as a source of standard integrals. In particular the integral

$$\int [\mathrm{d}\phi][\mathrm{d}\lambda_3][\mathrm{d}v_3] \exp\left(\mathrm{i}\int_{\tau_i}^{\tau_f} \mathrm{d}\tau(\dot{\lambda}_3 v_3 + \phi F(\lambda_3)\right)$$

can be rewritten as

$$\lim_{n\to\infty} \int \left(\prod_{j=0}^{2n} \mathrm{d}\phi_j\right)\left(\prod_{j=0}^{2n} \mathrm{d}\lambda_3^j\right)\left(\prod_{j=1}^{2n} \mathrm{d}v_3^j\right) \exp\left(\mathrm{i}\sum_{j=0}^{2n} [(\lambda_3^{j+1} - \lambda_3^j)v_3^j + \Delta\phi_j F_j(\lambda_3^j)]\right).$$

Expanding $F_j(\lambda_3^j)$ as $F_j(\lambda_3^j) = a_j + b_j\lambda_3^j$, where a_j is fermonic and b_j is bosonic, and then integrating, gives us

$$\lim_{n\to\infty} \int \mathrm{d}\lambda_3^0 \lambda_3^0 \Delta^{2n+1} \sum_{j=0}^{2n} \prod_{\substack{i=0\\ i\neq j}}^{2n} a_i b_j.$$

Eventually we obtain for Z

$$Z \sim \prod_{a=\alpha}^{\beta} (\bar{\theta}_a^{\mathrm{F}} - \bar{\theta}_a^{\mathrm{I}})(\tau_{\mathrm{f}} - \tau_{\mathrm{i}})^2 \int_{-\infty}^{\infty} \mathrm{d}\lambda_1^0 \delta(\lambda_1^0(\tau_{\mathrm{f}} - \tau_{\mathrm{i}})) \lim_{n\to\infty} \int \mathrm{d}\lambda_3^0 \lambda_3^0 \Delta^{2n+1} \sum_{j=0}^{2n} \prod_{\substack{i=0\\ i\neq j}}^{2n} a_i b_j$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}p_0^\mu \exp[\mathrm{i}(x_\mu^{\mathrm{F}} - x_\mu^{\mathrm{I}}) p_0^\mu] \int_{-\infty}^{\infty} \mathrm{d}\lambda_2^0 \exp[\mathrm{i}p_\mu^0 p_0^\mu (\tau_{\mathrm{f}} - \tau_{\mathrm{i}})\lambda_2^0] \tag{30}$$

where the index 0 indicates a value of a field which is independent of τ.

In terms of the Teichmuller parameters c_i (30) becomes

$$Z \sim \prod_{a=\alpha}^{\beta} (\bar{\theta}_a^{\mathrm{F}} - \bar{\theta}_a^{\mathrm{I}}) \int_{-\infty}^{\infty} \mathrm{d}c_1 \delta(c_1) \int \mathrm{d}c_3 c_3 \int_{-\infty}^{\infty} \mathrm{d}p_0^\mu \exp[\mathrm{i}(x_\mu^{\mathrm{F}} - x_\mu^{\mathrm{I}}) p_0^\mu]$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}c_2 \exp(\mathrm{i}p_\mu^0 p_0^\mu c_2) \tag{31}$$

where the undetermined constant

$$\lim_{n\to\infty} \Delta^{2n+1} \sum_{j=0}^{2n} \prod_{\substack{i=0\\ i\neq j}}^{2n} a_i b_j$$

has been absorbed into the normalisation (since different values of this constant lead to the same covering of Teichmuller space). So the gauge choice (27) is a good gauge choice; it results in a single covering of Teichmuller space.

The correct form of Z is obtained by restricting our integrals to a single cover of moduli space. In this case

$$\text{moduli space} = \frac{\{\lambda_i(\tau)\}}{\{\text{all diffeomorphisms}\}}. \tag{32}$$

We can see that our gauge fixing is not complete since we have not taken into account the disconnected diffeomorphisms which reverse the world line's orientation and interchange its end points. These leave the action invariant. Restricting our integrals to moduli space quotients out this extra symmetry. This disconnected diffeomorphism, which can be expressed as $\tau_{\mathrm{i}} \leftrightarrow \tau_{\mathrm{f}}$, $\mathrm{d}\tau \to -\mathrm{d}\tau$, maps c_i onto $-c_i$. So the modular group is $\mathbb{Z}_2$ and moduli space is $(\mathbb{R}^2 \times g)/\mathbb{Z}_2$ such that

$$Z \sim \prod_{a=\alpha}^{\beta} (\bar{\theta}_a^{\mathrm{F}} - \bar{\theta}_a^{\mathrm{I}}) \int_{-\infty}^{\infty} \mathrm{d}p_0^\mu \exp[\mathrm{i}(x_\mu^{\mathrm{F}} - x_\mu^{\mathrm{I}}) p_0^\mu] \int_{0}^{\infty} \mathrm{d}c_2 \exp(\mathrm{i}p_0^\mu p_\mu^0 c_2) \tag{33}$$

is the required superparticle propagator. Carrying out the c_2 integration we get a Feynman-like propagator for the superparticle:

$$Z \sim \prod_{a=\alpha}^{\beta} (\bar{\theta}_a^{\mathrm{F}} - \bar{\theta}_a^{\mathrm{I}}) \int_{-\infty}^{\infty} \mathrm{d}p_0^\mu \frac{\exp[\mathrm{i}(x_\mu^{\mathrm{F}} - x_\mu^{\mathrm{I}}) p_0^\mu]}{\mathrm{i}(p_0^\mu p_\mu^0 + \mathrm{i}\varepsilon)} \tag{34}$$

where ε is a positive parameter.

It was mentioned above that for $D=2$ the superparticle Lagrangian (1) describes a theory in an odd-dimensional phase space. The propagator (34) was constructed by applying the recipe given by Fradkin and Fradkina [10]. If we had eliminated the single redundant variable from our calculations before implementing the FV theorem there would not have been any obvious way to construct the path integral. Since second-class constraints are regarded as strong equations (as opposed to weak equations) this would, however, have been an equally valid way to approach the problem of quantising the system. The propagator (34) can, however, be used to calculate transition amplitudes between physical states just as if it were a transition kernel (see the discussion below and also Teitelboim [14]). Henneaux and Teitelboim [15] refer to propagators such as (34) as symbols of the evolution operator. It is shown below for the D-dimensional case that a propagator of the form of (34) maps (un)physical states into (un)physical states. The similarity between the heat kernel obtained from (34)

$$h(x_\mu, \bar{\theta}_a) \sim \lim_{\substack{\bar{\theta}^{\mathrm{F}}_a \to \bar{\theta}^{\mathrm{I}}_a \\ x^{\mathrm{F}}_\mu \to x^{\mathrm{I}}_\mu}} Z(x^{\mathrm{F}}_\mu, \bar{\theta}^{\mathrm{F}}_a, x^{\mathrm{I}}_\mu\, \bar{\theta}^{\mathrm{I}}_{\mathrm{a}}) \tag{35}$$

and the Witten index in two-dimensional superspace [16]

$$\lim_{\substack{\beta \to 0 \\ x \to x' \\ \theta \to \theta' \\ \bar{\theta} \to \bar{\theta}'}} \operatorname{Tr} \Gamma^3 \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \exp(-\mathrm{i}px' - \beta H + \mathrm{i}px) \prod_{a=1}^{2} \delta(\theta'_a - \theta_a)\delta(\bar{\theta}'_a - \bar{\theta}_a) \tag{36}$$

should also be noted.

3. The propagator for $D = 2, 3, 4$ modulo 8

Moving to the calculation of the superparticle propagator in spacetimes of D dimensions we find that in spaces of $D = 4$ or more we no longer have to deal with the difficulties of odd-dimensional phase spaces. In D dimensions we have

$$\operatorname{rank}(\not{p}C) \approx \tfrac{1}{2} \dim(\not{p}C) = \tfrac{1}{2} 2^{N/2} \tag{37}$$

where $N = D$ for even D and $N = (D-1)$ for odd D. So there will exist a separation of the components of T_{3a} into an equal number of first-class (T_A, $A = 3, \ldots, 2 + \frac{1}{2}2^{N/2}$) and second-class ($T_\alpha$, $\alpha = 3 + \frac{1}{2}2^{N/2}, \ldots, 2 + 2^{N/2}$) constraints. Such a separation will be achieved as follows:

$$A_{ab} T_{3b} = \begin{Bmatrix} T_A \\ T_\alpha \end{Bmatrix} \approx 0 \tag{38}$$

where A_{ab} is an invertible matrix of constant coefficients. The generalisation of (14) will be

$$\{A, B\}_{\mathrm{DB}} = \{A, B\}_{\mathrm{PB}} - \{A, T_\alpha\}_{\mathrm{PB}} \{T_\alpha, T_\beta\}^{-1}_{\mathrm{PB}} \{T_\beta, B\}_{\mathrm{PB}}. \tag{39}$$

As before, we now regard $T_\alpha = 0$ as strong equations and carry out BFV quantisation in terms of the first-class constraints $T_A \approx 0$. Our extended phase space is

$$\{x^\mu, p^\mu, e, \pi_e, \bar{\theta}_a, \rho_a, \lambda_i, v_i, \eta_i, \bar{\eta}_i, \zeta_i, \bar{\zeta}_i\} \qquad i = 1, 2, 3, \ldots, 2 + \tfrac{1}{2}2^{N/2} \tag{40}$$

and we have the constraints $G_u = (v_i, T_i)$. The Dirac bracket algebra of the constaints is

$$\{G_u, G_v\}_{\mathrm{DB}} = U^w_{uv} G_w = \delta_{u, A+2+(1/2)2^{N/2}} \delta_{v, B+2+(1/2)2^{N/2}} Y^w_{AB} G_w \delta_{w, i+2+(1/2)2^{N/2}}. \tag{41}$$

A separation of the components of T_{3a} such that

$$Y^w_{AB} = (0, 0, \ldots, Y^{4+(1/2)2^{N/2}}_{AB}, \ldots, 0, 0) \qquad Y^{4+(1/2)2^{N/2}}_{AB} = X_{AB}(p) \tag{42}$$

will always be possible since $\{T_A, T_B\}_{DB}$ is strongly equal to some function of $\not p$ but weakly vanishing and therefore also strongly equal to a linear combination of the first-class constraints. As before, we can now write down the BRST generator Ω

$$\Omega = v_i\zeta_i + T_i\eta_i + \tfrac{1}{2}\bar{\eta}_2 X_{AB}\eta_B\eta_A \qquad i = 1, 2, \ldots, 2+\tfrac{1}{2}2^{N/2} \tag{43}$$

and again check that $\{\Omega, \Omega\}_{DB} = 0$.

The BFV path integral is now

$$Z_\Psi \sim \int [dx^\mu][dp^\mu][de][d\pi_e][d\bar{\theta}_a][d\rho_a][d\lambda_i][dv_i][d\eta_i][d\bar{\eta}_i][d\zeta_i][d\bar{\zeta}_i]|\det\{T_\alpha, T_\beta\}_{PB}|^{1/2}$$
$$\times \prod_\alpha \delta[T_\alpha] \exp\left(i \int_{\tau_i}^{\tau_f} d\tau \{\dot{x}_\mu p^\mu + \dot{e}\pi_e + \dot{\theta}_a\rho_a + \dot{\lambda}_i v_i + \dot{\eta}_i\bar{\eta}_i + \dot{\zeta}\bar{\zeta}_i - \{\Psi, \Omega\}_{DB}\}\right)$$
$$i = 1, 2, \ldots, 2+\tfrac{1}{2}2^{N/2} \tag{44}$$

which is an obvious generalisation of (21). We have the boundary conditions (20) but with $i = 1, 2, \ldots, 2+\frac{1}{2}2^{N/2}$ and make suitable generalisations of (22) and (23). Making the gauge choice

$$\Psi = \bar{\eta}_1\lambda_1 + \bar{\eta}_2\lambda_2 + \frac{\phi_\alpha T_\alpha \bar{\eta}_2}{T_2} + \phi_\alpha F(\lambda_{\alpha-(1/2)2^{N/2}})\bar{\zeta}_1 - \xi_\alpha(i\pi|\{T_\alpha, T_\beta\}^{-1}_{PB}| - \delta_{\alpha\beta}v_1^2)\frac{\xi_\beta}{\pi_e}\bar{\eta}_1 \tag{45}$$

with the change of variables (28) gives, in the limit $\gamma \to 0$,

$$Z \sim \int [dx^\mu][dp^\mu][de][d\pi_e][d\bar{\theta}_a][d\rho_a][d\lambda_i][dv_i][d\eta_i][d\bar{\eta}_i][d\zeta_i][d\bar{\zeta}_i]$$
$$\times \prod_A \delta[F(\lambda_A)] \exp\left(i \int_{\tau_i}^{\tau_f} d\tau \{\dot{x}_\mu p^\mu + \dot{e}\pi_e + \dot{\theta}_a\rho_a + \dot{\lambda}_i v_i\right.$$
$$\left. + \dot{\eta}_i\bar{\eta}_i + \dot{\zeta}_i\bar{\zeta}_i - \bar{\eta}_1\zeta_1 + \lambda_1\pi_e - \bar{\eta}_2\zeta_2 + \lambda_2 p_\mu p^\mu\}\right). \tag{46}$$

All these integrals can be carried out by analogy with the $D = 2$ case and we find the following integral over Teichmuller space

$$Z \sim \prod_a (\bar{\theta}_a^F - \bar{\theta}_a^I) \int_{-\infty}^{\infty} dc_1\, \delta(c_1) \prod_A \int dc_A\, c_A \int_{-\infty}^{\infty} dp_\mu^0 \exp[i(x_F^\mu - x_I^\mu)p_\mu^0]$$
$$\times \int_{-\infty}^{\infty} dc_2 \exp(ip_\mu^0 p_0^\mu c_2) \tag{47}$$

where the c_i are Teichmuller parameters defined as above. Restricting the region of integration to a single cover of moduli space gives

$$Z \sim \prod_a (\bar{\theta}_a^F - \bar{\theta}_a^I) \int_{-\infty}^{\infty} dp_\mu^0 \frac{\exp[i(x_F^\mu - x_I^\mu)p_\mu^0]}{i(p_\mu^0 p_0^\mu + i\varepsilon)}. \tag{48}$$

This is a standard integral (see Gel'fand and Shilov [7]) which for the specific metric

$$g^{\mu\nu} = (+1, -1, -1, \ldots, -1) \tag{49}$$

is given by

$$Z \sim \prod_{a=1}^{2^{N/2}} (\bar{\theta}_a^F - \bar{\theta}_a^I) \frac{(\exp - i(D-1)\pi/2)2^{D-2}\pi^{D/2}\Gamma(D/2-1)}{((x_F^\mu - x_I^\mu)^2 - i\varepsilon)^{D/2-1}}. \tag{50}$$

4. Discussion

In the physically interesting cases of $D=4$ and $D=10$ we do not have to deal with an odd-dimensional phase space but (48) still cannot be regarded as a transition amplitude between observable states. The propagator we have obtained is an amplitude between states in superspace. It is not complex valued, as should be the case for a quantum mechanical amplitude. Function (48) is a Grassman-valued function but can, however, be used to calculate transition amplitudes by identifying it as a transition kernel and expressing a complex-valued amplitude $\langle\chi_F|\psi_I\rangle$ as

$$\langle\chi_F|\psi_I\rangle=\int d\bar{\theta}_a^F\, d\bar{\theta}_a^I\, dx_\mu^F\, dx_\mu^I \chi_F^* Z(\bar{\theta}_a^F, x_\mu^F, \bar{\theta}_a^I, x_\mu^I)\psi_I. \tag{51}$$

See Pugh [18] for a discussion of this point.

We can demonstrate the operation of our propagator by studying its effect on physical states. In an operator approach to the quantisation of constrained systems physical states are those which are annihilated by the first-class constraints and the second-class constraints are taken as relationships between quantum operators. Applying these considerations to the superparticle, let us define a wavefunction $\chi(\bar{\theta}_a, x_\mu)$; then, if

$$T_i\chi=0 \qquad i=1,\ldots,2+\tfrac{1}{2}2^{N/2} \tag{52}$$

χ describes a physical state of the system. Consider a state $\chi(\bar{\theta}_a^I, x_\mu^I)$, then our propagator gives us the state $\chi(\bar{\theta}_a^F, x_\mu^F)$:

$$\chi(\bar{\theta}_a^F, x_\mu^F)=\int d\bar{\theta}_a^I\, dx_\mu^I Z(\bar{\theta}_a^F, x_\mu^F, \bar{\theta}_a^I, x_\mu^I)\chi(\bar{\theta}_a^I, x_\mu^I). \tag{53}$$

We find that

$$T_i\chi(\bar{\theta}_a^F, x_\mu^F)=\int d\bar{\theta}_a^I\, dx_\mu^I Z(\bar{\theta}_a^F, x_\mu^F, \bar{\theta}_a^I, x_\mu^I) T_i\chi(\bar{\theta}_a^I, x_\mu^I) \tag{54}$$

so we see that

$$T_i\chi(\bar{\theta}_a^I, x_\mu^I)=0 \Leftrightarrow T_i\chi(\bar{\theta}_a^F, x_\mu^F)=0 \tag{55}$$

and

$$T_i\chi(\bar{\theta}_a^I, x_\mu^I)\neq 0 \Leftrightarrow T_i\chi(\bar{\theta}_a^F, x_\mu^F)\neq 0 \tag{56}$$

i.e. our propagator maps (un)physical states into (un)physical states.

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Appendix

If path integrals are to be understood as the continuous limit of a discrete set of one-dimensional integrals then Z_Ψ as given in (21) can be rewritten in terms of

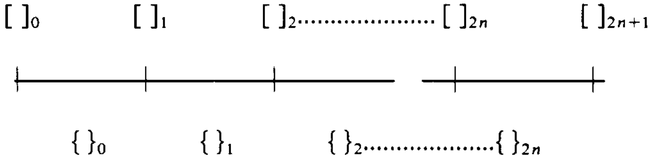

Figure 1. One-dimensional lattice for functional integral variables. [ ] represents the variables $[x_\mu, e, \bar\theta_a, v_i, \eta_i, \bar\zeta_i]$ and { } represents $\{p_\mu, \pi_e, \rho_a, \lambda_i, \bar\eta_i, \zeta_i\}$.

discretised phase space variables. Set up a lattice of $2n+2$ sites with an interval Δ between neighbouring sites. (The lattice is chosen to have $2n+2$ sites so that all parts of the measure have a definite Grassman parity as $n \to \infty$.) Attach the extended phase space variables (21) to the lattice. The variables subject to boundary conditions are placed on the sites with their conjugate variables placed between sites, as shown in figure 1. The path integral (21) is then expressible in terms of the lattice variables as

$$
\begin{aligned}
Z_\psi \sim \lim_{n\to\infty} \int & \prod_{j=1}^{2n} \mathrm{d}x^j_\mu \prod_{j=0}^{2n} \mathrm{d}p^j_\mu \prod_{j=1}^{2n} \mathrm{d}e^j \prod_{j=0}^{2n} \mathrm{d}\pi^j_e \prod_{j=1}^{2n} \mathrm{d}\bar\theta^j_a \prod_{j=0}^{2n} \mathrm{d}\rho^j_a \prod_{j=0}^{2n} \mathrm{d}\lambda^j_i \\
& \prod_{j=1}^{2n} \mathrm{d}v^j_i \prod_{j=1}^{2n} \mathrm{d}\eta^j_i \prod_{j=0}^{2n} \mathrm{d}\bar\eta^j_i \prod_{j=0}^{2n} \mathrm{d}\zeta^j_i \prod_{j=1}^{2n} \mathrm{d}\bar\zeta^j_i \prod_{j=0}^{2n} \left(\frac{|4\mathrm{i}p^1_j|}{\Delta}\right)^{1/2} \prod_{j=0}^{2n} \delta \\
& \times[-\rho^j_\alpha + \rho^j_\beta + \tfrac{1}{2}(\bar\theta^{j+1}_\alpha + \bar\theta^j_\alpha)(-p^j_0 + \mathrm{i}p^j_1) + \tfrac{1}{2}(\bar\theta^{j+1}_\beta + \bar\theta^j_\beta)(-p^j_0 - \mathrm{i}p^j_1)] \\
& \times \exp\Bigg(\mathrm{i} \sum_{j=0}^{2n} [(x^{j+1}_\mu - x^j_\mu)p^\mu_j + (e^{j+1} - e^j)\pi^j_e + (\bar\theta^{j+1}_a - \bar\theta^j_a)\rho^j_a - \lambda^j(v^{j+1}_i - v^j_i) \\
& + (\eta^{j+1}_i - \eta^j_i)\bar\eta^j_i - \zeta^j_i(\bar\zeta^{j+1}_i - \bar\zeta^j_i) - \Delta\{\Psi, \Omega\}^j_{\mathrm{DB}}]\Bigg). \qquad (57)
\end{aligned}
$$

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